

Some Results on Zero-Sum Sequences in Z_p^3

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Abstract

Kemnitz Conjecture [9] states that if we take a sequence of elements in Z_p^2 of length $4p - 3$, p is a prime number, then it has a subsequence of length p , whose sum is 0 modulo p . It is known that in Z_p^3 to get a similar result we have to take a sequence of length atleast $9p - 8$. In this paper we will show that if we add a condition on the chosen sequence, then we can get a good upper and a lower bound for which similar results hold.

Introduction

Denoting by $s_k(Z_n^d)$ the smallest integer t such that any set of t lattice-points in the d -dimensional Euclidean space contains a subset of cardinality kn , the sum of whose elements is divisible by n , it was first proved by Erdős, Ginzburg and Ziv [4], that $s_1(Z_n) = 2n - 1$. Kemnitz' Conjecture $s_1(Z_n^2) = 4n - 3$ was open for about twenty years and was proved by Reiher in [9] after a series of results by Gao [5], Rónyai [10] and others. Up to now the best general bounds for odd primes p and higher dimensions d are $s_1(Z_p^d) \geq 1.125^{\lfloor \frac{d}{3} \rfloor} 2^d (p - 1) + 1$, by Elsholtz [3], and $s_1(Z_p^d) \leq (cd \log d)^d p$ by Alon and Dubiner [1], where c is a constant. They conjectured that $s_1(Z_p^d) \leq c^d p$. In 2001, Elsholtz [3] showed that $s_1(Z_n^3) \geq 9n - 8$. Bhowmik and Schalge-Puchta [2] proved that $s_1(Z_p^3) = 9p - 8$ for $p \rightarrow \infty$, p is a prime number. Hence, it is natural to ask whether $s_1(Z_p^3) = 9p - 8$ for all p . We are as yet unable to answer this question. However, we study the constant $s_I(Z_p^3)$ for certain sequences I . Kubertin [8], Gao, Thangadurai [6] and Geroldinger, Gryniewicz, Schmid [7] have studied some properties of these kind of constants. Gao, Thangadurai

[6] studied this constant for groups $G \cong Z_n^d$ when $d=3$ or 4 and proved that $s_k(Z_p^3) = kp + 3p - 3$ for every $k \geq 4$, where p is a prime number. Kubertin [8] further extended this result by proving that $s_k(Z_q^3) = (k+3)q - 3$ for $k \geq 3$ and q be a prime power of $p > 3$ and $s_k(Z_q^4) = (k+4)q - 4$ for $k \geq 4$ and $p \geq 7$, p is a prime number and q is a prime power of p . She conjectured that for positive integers $k \geq d$ and n we have $s_k(Z_n^d) = (k+d)n - d$ and proved that the conjecture holds for $s_{np}(Z_q^d)$ where p is a prime number and q is a power of p . Geroldinger, Griynkiewicz, Schmid [7] defined for a finite abelian group G and a positive integer d , $s_{d\mathbb{N}}(G)$ to be the smallest integer $l \in \mathbb{N}_0$ such that every sequence S over G of length $|S| \geq l$ has a non-empty zero-sum subsequence T of length $|T| \equiv 0 \pmod{d}$. They showed that, Let $d \in \mathbb{N}$ and let $n = \exp(G)$. Suppose G is cyclic. Then $s_{d\mathbb{N}}(G) = \text{lcm}(n, d) + \gcd(n, d) - 1$.

They also determined $s_{d\mathbb{N}}(G)$ for all $d \geq 1$ when G has rank at most two and, under mild conditions on d , and obtained precise values in the case of p -groups. Continuing on this line in this paper we give an upper bound and a lower bound for $s_1(Z_n^3)$ of a particular kind of sequences. We have used the idea of ‘lifting of an equation’ (Explained Later) by Reiher [9] for studying some properties of the sequences in Z_p^3 and have generalized the function used in Ronyai’s Method [8] to prove one of our theorems. We have used the ‘Polynomial Methods’ to study the zero-sum properties of the sequences in Z_p^3 . We must note that $s_1(Z_n^d)$ is a completely multiplicative function of n . Here, we will prove our results for the prime numbers p which essentially proves for the other integers n also. Throughout the text p denotes a prime number and n stands for any integer.

Main Results and Two Applications of Polynomial Methods

Main Theorems

If I is a sequence of elements in Z_p^3 , then $N^{kp}(I)$ denotes the number of subsequences of I of length kp , whose sum is 0 modulo p . By $a \equiv b$, we mean $a \equiv b \pmod{p}$. In this paper, we will prove the following three theorems:

Theorem 1. *Let J be a sequence of elements in Z_p^3 with $|J| = 7p - 3$. Let, for all $I \subset J$ with $|I| = 4p - 3$, $N^{2p}(I) \equiv c \pmod{p}$, where c is a fixed number, then J has a subsequence of length p , whose sum is 0 modulo p when $p > 7$.*

Theorem 2. *There is a sequence of elements J in Z_p^3 with $|J| = 4p - 4 + \frac{p-1}{2}$ such that for all $I \subset J$ with $|I| = 4p - 3$, $N^{2p}(I) \equiv 0 \pmod{p}$, and J does not*

have any subsequence of length p , whose sum is 0 modulo p .

We also have investigated the case when $N^p(J) = 0$, J is a sequence of Z_p^3 . For such a sequence J with $|J| = 9p - 3$, we checked whether it has the subsequences of length ip , whose sum is 0 modulo p , for $2 \leq i \leq 8$. And we have come up with the result :

Theorem 3. Let $J = \{(a_i, b_i, c_i) , 1 \leq i \leq (9p - 3)\}$ be a sequence in Z_p^3 . We have either $N^p(J) > 0$ or six of the $N^{ip}(J)$'s are not congruent to 0, $2 \leq i \leq 8$.

We need the following definitions to prove the above theorem.

Definition. We define $g(x) = x_1^{p-1} + \dots + x_{9p-3}^{p-1}$. And with the help of it we define,

$$S_i = \left(\binom{g(x)}{p} - i \right), Q_1 = \binom{a_1 x_1^{p-1} + \dots + a_{9p-3} x_{9p-3}^{p-1}}{p-1},$$

$$Q_2 = \binom{b_1 x_1^{p-1} + \dots + b_{9p-3} x_{9p-3}^{p-1}}{p-1}, Q_3 = \binom{c_1 x_1^{p-1} + \dots + c_{9p-3} x_{9p-3}^{p-1}}{p-1}.$$

We also define a new function $P_{18}(x) = \binom{g(x)-1}{p-1} Q_1 Q_2 Q_3 S_2 S_3 \dots S_7$ where only S_1, S_8 are missing. $P_{ij}(x)$ is defined similarly in which only S_i, S_j are missing.

Remark. "By lifting one of the above equations" follows the same technique that Reiher [9] used in his paper.

As an example, if J is a sequence in Z_p^2 of cardinality $3p - 3$, then by the polynomial method [9] we get the equation $1 - N^{p-1}(J) - N^p(J) + N^{2p-1}(J) + N^{2p}(J) \equiv 0$. Now, let X be a sequence of Z_p^2 of cardinality $4p - 3$. Then, we get:- $\sum \{1 - N^{p-1}(J) - N^p(J) + N^{2p-1}(J) + N^{2p}(J)\} \equiv 0$, where the sum is extended over all $J \subset X$ of cardinality $3p - 3$. Analysing the number of times each subsequence is counted we obtain $\binom{4p-3}{3p-3} - \binom{3p-2}{2p-2} N^{p-1}(X) - \binom{3p-3}{2p-3} N^p(X) + \binom{2p-2}{p-2} N^{2p-1}(X) + \binom{2p-3}{p-3} N^{2p}(X) \equiv 0$. After the reduction we get, $3 - 2N^{p-1}(X) - 2N^p(X) + N^{2p-1}(X) + N^{2p}(X) \equiv 0$. This equation will be called a lifting of $1 - N^{p-1}(J) - N^p(J) + N^{2p-1}(J) + N^{2p}(J) \equiv 0$. \square

Existence of a $2p$ -Zero Sum Sequence in Z_p^3

Statement. If I is a sequence of elements in Z_p^3 and $|I| = 6p - 3$, then I has a subsequence of length $2p$, whose sum is congruent to 0 modulo p .

Proof. If $N^{4p}(I) > 0$ then there exists a subsequence J of I such that $|J| = 4p$ and $\sum J \equiv 0$. Define $K = J - x$ where x is any element of J . Using the

technique used by Reiher [9] we get the equation, $1 - N^p(K) + N^{2p}(K) - N^{3p}(K) \equiv 0$. So, one of the $N^p(K), N^{2p}(K), N^{3p}(K)$ has to be non-zero. And we get that:- either J has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. Now let $|J| = 5p - 3$. Then we have this equation $4 - 3N^p(J) + 2N^{2p}(J) - N^{3p}(J) \equiv 0$ by lifting the equation [9] $1 - N^p(I) + N^{2p}(I) - N^{3p}(I) \equiv 0$ where $|I| = 4p - 3$. We also have the equation $1 - N^p(J) + N^{2p}(J) - N^{3p}(J) + N^{4p}(J) \equiv 0$. Our claim is that either J has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. If not, then we get $N^{3p}(J) \equiv 4$ and therefore we obtain $N^{4p}(J) \equiv 3$ and hence either J has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$.

Finally, if $|L| = 6p - 3$, then $6p - 3 > 5p - 3$, so we have either L has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. Let $N^{2p}(L) = 0$. So, we have $T \subset L$ such that $|T| = p$ and $\sum T \equiv 0$. Now $|L - T| = 5p - 3$ and by repeating the previous argument we will get another p sequence whose sum is $0 \pmod p$, call it V . Then $V \cup T$ is a $2p$ sequence whose sum is 0 . \square

Existence of a $3p$ -Zero Sum Sequence in Z_p^3

Statement. *If J is a sequence of Z_p^3 such that $|J| = 7p - 3$, then J has a subsequence of length $3p$, whose sum is congruent to 0 modulo p .*

Proof. Kubertin [8] proved that this bound can be reduced to $6p - 3$. But, by only using the polynomial method [9] we get an upper bound equals to $7p - 3$.

Firstly, we show that if $|I| = 5p$ and $\sum I \equiv 0$ then either I has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. Let, $H = I - x$ where x is an element of I , then we have the equation: $1 - N^p(H) + N^{2p}(H) - N^{3p}(H) + N^{4p}(H) \equiv 0$. So, if $N^p(I) = 0, N^{2p}(I) = 0$ then either $N^{3p}(H) > 0$ or $N^{4p}(H) > 0$ and which contradicts the assumption. Now, we show that if $|J| = 6p - 3$ then either J has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. If not, then we have these two equations,

$$1 - N^{3p}(J) + N^{4p}(J) - N^{5p}(J) \equiv 0.$$

and $5 - 2N^{3p}(J) + N^{4p}(J) \equiv 0$ by lifting the equation: $1 - N^p(L) + N^{2p}(L) - N^{3p}(L) + N^{4p}(L) \equiv 0$ where $|L| = 5p - 3$. And we get the equation $-4 + N^{3p}(J) - N^{5p}(J) \equiv 0$. So, either $N^{3p}(J) > 0$ or $N^{5p}(J) > 0$ and hence the result follows. Now, if $|J| = 7p - 3$ and J has either a subsequence of

length p whose sum is $0 \pmod p$ or it has a subsequence of length $3p$ whose sum is $0 \pmod p$ then it is okay. Otherwise, we will have these equations :

1. $6+4N^{2p}(J)+2N^{4p}(J)-N^{5p}(J) \equiv 0$ by lifting the equation : $1-N^p(H)+N^{2p}(H)-N^{3p}(H)+N^{4p}(H)-N^{5p}(H) \equiv 0$ where $|H| = 6p-3$.
2. $15+6N^{2p}(J)+N^{4p}(J) \equiv 0$ by lifting : $1-N^p(H)+N^{2p}(H)-N^{3p}(H)+N^{4p}(H) \equiv 0$ where $|H| = 5p-3$.
3. $5+N^{2p}(J) \equiv 0$ by lifting : $1-N^p(H)+N^{2p}(H)-N^{3p}(H) \equiv 0$ where $|H| = 4p-3$.

And all these gives us the new equation $N^{5p}(J) \equiv 16$. And we have seen that $|T| = 5p$, $\sum T \equiv 0$ implies either T has a subsequence of length p whose sum is $0 \pmod p$ or it has a subsequence of length $2p$ whose sum is $0 \pmod p$. But as we have assumed that $N^p(T) = 0$, therefore there exists $M \subset T$ such that $|M| = 2p$ and $\sum M \equiv 0$. And $T - M$ satisfies our condition. \square

Corollary 1. *It is also clear that if I is a sequence of Z_p^3 with $|I| = 8p-3$ where p is a prime number, then I has a subsequence of length $4p$ whose sum is $0 \pmod p$ and if $|I| = 9p-3$ then I has subsequence of length $5p$ whose sum is $0 \pmod p$.*

Proofs

Results On Z_p^3 Sequences leading to the proofs

We have used this result in several occasions without mentioning it, $\binom{kp-c}{rp-c} \equiv \frac{(k-1)\dots r}{(k-r)!}$ and we are assuming that $p > 7$ for the rest of the text.

Let J be a sequence of Z_p^3 with $|J| = 9p-3$. Then we have the following equations by using the technique used in [9] :

1. $1-N^p(J)+N^{2p}(J)-N^{3p}(J)+N^{4p}(J)-N^{5p}(J)+N^{6p}(J)-N^{7p}(J)+N^{8p}(J) \equiv 0$.
2. $1-N^p(J_1)+N^{2p}(J_1)-N^{3p}(J_1)+N^{4p}(J_1)-N^{5p}(J_1)+N^{6p}(J_1)-N^{7p}(J_1) \equiv 0$ where J_1 is a subsequence of J with $|J_1| = 8p-3$.
3. $1-N^p(J_2)+N^{2p}(J_2)-N^{3p}(J_2)+N^{4p}(J_2)-N^{5p}(J_2)+N^{6p}(J_2) \equiv 0$ where J_2 is a subsequence of J with $|J_2| = 7p-3$.

4. $1 - N^p(J_3) + N^{2p}(J_3) - N^{3p}(J_3) + N^{4p}(J_3) - N^{5p}(J_3) \equiv 0$ where J_3 is a subsequence of J with $|J_3| = 6p - 3$.
5. $1 - N^p(J_4) + N^{2p}(J_4) - N^{3p}(J_4) + N^{4p}(J_4) \equiv 0$ where J_4 is a subsequence of J with $|J_4| = 5p - 3$.
6. $1 - N^p(J_5) + N^{2p}(J_5) - N^{3p}(J_5) \equiv 0$ where J_5 is a subsequence of J with $|J_5| = 4p - 3$.

By lifting the above equations we get :

1. $56 - 21N^p(J) + 6N^{2p}(J) - N^{3p}(J) \equiv 0$ after lifting the above equation (6).
2. $70 - 35N^p(J) + 15N^{2p}(J) - 5N^{3p}(J) + N^{4p}(J) \equiv 0$ after lifting the above equation (5).
3. $56 - 35N^p(J) + 20N^{2p}(J) - 10N^{3p}(J) + 4N^{4p}(J) - N^{5p}(J) \equiv 0$ after lifting the above equation (4).
4. $28 - 21N^p(J) + 15N^{2p}(J) - 10N^{3p}(J) + 6N^{4p}(J) - 3N^{5p}(J) + N^{6p}(J) \equiv 0$ after lifting the above equation (3).
5. $8 - 7N^p(J) + 6N^{2p}(J) - 5N^{3p}(J) + 4N^{4p}(J) - 3N^{5p}(J) + 2N^{6p}(J) - N^{7p}(J) \equiv 0$ after lifting the above equation (2).
6. $1 - N^p(J) + N^{2p}(J) - N^{3p}(J) + N^{4p}(J) - N^{5p}(J) + N^{6p}(J) - N^{7p}(J) + N^{8p}(J) \equiv 0$.

Now, assume that, $N^{2p}(J) \equiv k$ and $N^p(J) = 0$. Then, we can rewrite the above equations in the form of a matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 & 0 & 0 & 0 \\ 15 & -5 & 1 & 0 & 0 & 0 & 0 \\ 20 & -10 & 4 & -1 & 0 & 0 & 0 \\ 15 & -10 & 6 & -3 & 1 & 0 & 0 \\ 6 & -5 & 4 & -3 & 2 & -1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \\ N^{5p}(J) \\ N^{6p}(J) \\ N^{7p}(J) \\ N^{8p}(J) \end{pmatrix} = \begin{pmatrix} k \\ -56 \\ -70 \\ -56 \\ -28 \\ -8 \\ -1 \end{pmatrix}$$

And we get,

$$\begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \\ N^{5p}(J) \\ N^{6p}(J) \\ N^{7p}(J) \\ N^{8p}(J) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & -1 & 0 & 0 & 0 & 0 & 0 \\ 15 & -5 & 1 & 0 & 0 & 0 & 0 \\ 20 & -10 & 4 & -1 & 0 & 0 & 0 \\ 15 & -10 & 6 & -3 & 1 & 0 & 0 \\ 6 & -5 & 4 & -3 & 2 & -1 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} k \\ -56 \\ -70 \\ -56 \\ -28 \\ -8 \\ -1 \end{pmatrix} = \begin{pmatrix} k \\ 6k + 56 \\ 15k + 210 \\ 20k + 336 \\ 15k + 252 \\ 6k + 120 \\ k + 21 \end{pmatrix}$$

Among the numbers $k, 6k+56, 15k+210, 6k+120, k+21, 20k+336, 15k+252$, p cannot divide two of these numbers simultaneously, except $20k+336, 15k+252$ as $p > 7$. (i.e. if p divides $6k+56$ and $6k+120$, then p divides 64 but p is a prime number > 7 . So, it is not possible. The similar argument proves the claim.)

Corollary 2. *If J is a sequence in Z_p^3 with $|J| = 8p - 3$, $N^{2p}(J) \equiv l \pmod{p}$ and $N^p(J) = 0$, then we get*

$$\begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \\ N^{5p}(J) \\ N^{6p}(J) \\ N^{7p}(J) \end{pmatrix} = \begin{pmatrix} l \\ 5l + 35 \\ 10l + 105 \\ 10l + 122 \\ 5l + 52 \\ l + 1 \end{pmatrix} \text{ and if } p \text{ divides one of the numbers on the}$$

right hand side then it cannot divide the others if $p \neq 13, 17, 19, 47, 61$.

Corollary 3. *If J is a sequence in Z_p^3 with $|J| = 7p - 3$, $N^{2p}(J) \equiv m \pmod{p}$ and $N^p(J) = 0$, then we get*

$$\begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \\ N^{5p}(J) \\ N^{6p}(J) \end{pmatrix} = \begin{pmatrix} m \\ 4m + 20 \\ 6m + 45 \\ 4m + 36 \\ m + 10 \end{pmatrix} \text{ and if } p \text{ divides one of the numbers on the}$$

right hand side then it cannot divide the others.

Corollary 4. *If J is a sequence in Z_p^3 with $|J| = 6p - 3$, $N^{2p}(J) \equiv t \pmod{p}$ and $N^p(J) = 0$, then we get*

$$\begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \\ N^{5p}(J) \end{pmatrix} = \begin{pmatrix} t \\ 3t + 10 \\ 3t + 15 \\ t + 6 \end{pmatrix} \text{ and if } p \text{ divides one of the numbers on the right}$$

hand side then it cannot divide the others.

Corollary 5. *If J is a sequence in Z_p^3 with $|J| = 5p - 3$, $N^{2p}(J) \equiv r \pmod{p}$ and $N^p(J) = 0$ then we get*

$$\begin{pmatrix} N^{2p}(J) \\ N^{3p}(J) \\ N^{4p}(J) \end{pmatrix} = \begin{pmatrix} r \\ 2r + 4 \\ r + 3 \end{pmatrix} \text{ and if } p \text{ divides one of the numbers on the right}$$

hand side then it cannot divide the others.

Remark. *If $|I| = 9p - 3$, $N^p(J) = 0$ and $N^{2p}(J) \equiv c$, c is a fixed number $\forall J$ such that $|J| = 4p - 3, J \subset I$ then we have the following relations $5r \equiv 3t, 5r \equiv 2m, 7r \equiv 2l, 3k \equiv 14t$ and $3t \equiv 10c, 2m \equiv 10c, r \equiv 2c$. Here, t, r, m, l, k are the variables defined earlier.*

Application.

If J is a sequence in Z_p^3 with $|J| = 9p - 3$ and $N^p(J) = 0$, then J has a subsequence of length $6p$ whose sum is $0 \pmod{p}$ where p is a prime number > 7 and $p \neq 13, 17, 19, 47, 61$.

Proof. If $N^{6p}(J) = 0$, then by using the earlier calculations we get $N^{8p}(J) \neq 0$. So, there exists $I \subset J$ such that $|I| = 8p$ and $\sum I \equiv 0$. Let $M = I - \{x, y, z\}$. Here, $|M| = 8p - 3$. So, using the corollary1 we get that $N^{7p}(M) \neq 0$. So, there exist $L \subset M$ such that $|L| = 7p$ and $\sum L \equiv 0$. Now, $|I - L| = p$ and $\sum(I - L) \equiv 0$. But, it is a contradiction to our hypothesis. So, $N^{6p}(J) \neq 0$. \square

Proof Of The Main Results

Proof of Theorem1. Let there be a subsequence $I \subset J$, $|I| = 6p$ and $\sum I \equiv 0$. Now, if there exists a subsequence $K \subset I$ such that $|K| = 5p$ and $\sum K \equiv 0$ then we are done. Otherwise, $t \equiv -6 \Rightarrow 3t \equiv -18 \equiv 10c \Rightarrow 5c \equiv -9$, where $t = N^{2p}(I'), |I'| = 6p - 3$ and $I' \subset I$. Here, $N^{4p}(I') \equiv -3$. Let $I'' \subset I', |I''| = 4p, \sum I'' \equiv 0$. Let $L \subset I'', |L| = 4p - 3$. We get the equation, $1 + N^{2p}(L) - N^{3p}(L) \equiv 0 \Rightarrow 5 + 5c - 5N^{3p}(L) \equiv 0 \Rightarrow N^{3p}(L) \neq 0$ as $5c \equiv -9$. So, we have got a p zero-sequence. If we do not have a $6p$ zero sequence inside J , then $m \equiv -10$ ie $N^{5p}(J) \neq 0$ and $10c \equiv -20, r \equiv -4$, so we are done. \square

Proof of Theorem2. Consider the set $I = \{(0, 0, 0)^{p-1}, (1, 0, 0)^{p-1}, (0, 1, 0)^{p-1}, (0, 0, 1)^{p-1}, (1, 1, 1)^{\frac{p-1}{2}}\}$. Then, $|I| = 4p - 4 + \frac{p-1}{2}$. I neither has a p -zero-sum sequence of elements nor has a $2p$ -zero-sum sequence of elements. So, I satisfies the condition that $N^{2p}(I) \equiv 0$ but it does not have a p -zero-sum sequence. \square

Remark. So, $4p - 4 + \frac{p-1}{2}$ is a lower bound for this result.

Lemma. $\sum_{x \in Z_p^{9p-3}} P_{mn}(x) \equiv \sum_{x \in Z_p^{9p-3}} P_{rs}(x)$ for $m \neq n$, $r \neq s$.

Proof. We will prove that $\sum_{x \in Z_p^{9p-3}} P_{18}(x) \equiv \sum_{x \in Z_p^{9p-3}} P_{17}(x)$. And the rest can be derived from this :

$$\begin{aligned} \sum_{x \in Z_p^{9p-3}} P_{18}(x) &= \sum_{x \in Z_p^{9p-3}} \binom{g(x)-1}{p-1} Q_1 Q_2 Q_3 S_2 S_3 \dots S_7 \\ &= \sum_{x \in Z_p^{9p-3}} \binom{g(x)-1}{p-1} Q_1 Q_2 Q_3 S_2 \dots S_6 \left(\binom{g(x)}{p} - 7 \right) \\ &= \sum_{x \in Z_p^{9p-3}} \binom{g(x)-1}{p-1} Q_1 Q_2 Q_3 S_2 S_3 \dots S_6 \left(\binom{g(x)}{p} - 8 + 1 \right) \\ &= \sum_{x \in Z_p^{9p-3}} P_{17}(x) + \sum_{x \in Z_p^{9p-3}} \binom{g(x)-1}{p-1} Q_1 Q_2 Q_3 S_2 S_3 \dots S_6 \\ &= \sum_{x \in Z_p^{9p-3}} P_{17}(x) + \sum_{x \in Z_p^{9p-3}} R. \end{aligned}$$

We will calculate the value of R directly.

$$R = \sum_{1 \leq i \leq (9p-3)} a_{k_1 \dots k_{9p-3}} \prod x_i^{k_i(p-1)}, \text{ where } k_i \geq 0.$$

And $\sum_{x \in Z_p^{9p-3}} a_{k_1 \dots k_{9p-3}} \prod x_i^{k_i(p-1)} \equiv 0$ if one of the $k_i = 0$. Hence, R can be non-zero if $\text{degree}(R)$ is at least equal to $(9p-3)(p-1)$. But in our case $\text{degree}(R)$ is $(9p-4)(p-1)$ (degree of S_i is p for all i , and $\text{degree}\left(\binom{g(x)-1}{p-1}\right) = \text{degree}(Q_i) = p-1$). So, our claim is established. \square

Proof of Theorem3. These are some observations :

1. $P_{18}(x) \equiv 0$ if $g(x) \not\equiv 0 \pmod p$.
2. $P_{18}(x) \equiv 0$ if $\sum_{1 \leq i \leq (9p-3)} a_i x_i^{(p-1)} \not\equiv 0 \pmod p$.
3. $P_{18}(x) \equiv 0$ if $\sum_{1 \leq i \leq (9p-3)} b_i x_i^{(p-1)} \not\equiv 0 \pmod p$.
4. $P_{18}(x) \equiv 0$ if $\sum_{1 \leq i \leq (9p-3)} c_i x_i^{(p-1)} \not\equiv 0 \pmod p$.

Define, $m = \sum_{x \in Z_p^{9p-3}} P_{18}(x)$.

After calculating directly we get,

$$\sum_{x \in Z_p^{9p-3}} P_{18}(x) \equiv P_{18}(0) + (p-1)^p N^p(J) 6! + (p-1)^{8p} N^{8p}(J) 6!.$$

$$m - P_{18}(0) \equiv (p-1)^p N^p(J) 6! + (p-1)^{8p} N^{8p}(J) 6!.$$

Similarly, we can get a set of equations :

$$m - P_{18}(0) = a_1 N^p(J) + b_1 N^{8p}(J).$$

$$m - P_{17}(0) = a_2 N^p(J) + b_2 N^{7p}(J). \quad [\text{As, } \sum_{x \in Z_p^{9p-3}} P_{18}(x) = \sum_{x \in Z_p^{9p-3}} P_{17}(x) \text{ by Lemma 3 }]$$

...

$$m - P_{12}(0) = a_7 N^p(J) + b_7 N^{2p}(J). \quad [\text{As, } \sum_{x \in Z_p^{9p-3}} P_{18}(x) = \sum_{x \in Z_p^{9p-3}} P_{12}(x) \text{ by Lemma 3 }]$$

where a_i, b_j are integers. And the values of the $P_{1j}(0)$ are different. So, the result follows. \square

If we can show that $N^{ip}(J) > 0$, $2 \leq i \leq 8$ for a sequence J of Z_p^3 where $|J| = 9p - 3$ by using the techniques used to prove Theorem 3, then it also proves that J has a zero-sum of length p , which proves that $s_1(Z_p^3) = 9p - 3$. And, this is a very good upper bound for $s_1(Z_p^3)$. This is in the line of the following conjecture,

Conjecture. $s_1(Z_p^3) = 9p - 8$ [2].

If we can say more about the sum $m = \sum_{x \in Z_p^{9p-3}} P_{18}(x)$, then it will help us to prove our claim.

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